Differential Geometry and its Applications 4 (1994) 45–69
North-Holland

Non-homogeneous relatives of symmetric spaces

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Received 8 October 1992

Abstract: For every fixed Riemannian symmetric space $(\tilde{M},\tilde{g})$ we determine explicitly all *locally non-homogeneous* Riemannian spaces which have, at all points, the same curvature tensor as $(\tilde{M},\tilde{g})$. For this purpose, we describe explicitly all parabolically foliated semi-symmetric spaces in the sense of Z.I. Szabó.

Keywords: Semi-symmetric spaces, curvature homogeneous manifolds.

MS classification: 53B20, 53C12, 53C20, 53C21, 53C35.

1. Introduction

Let (M,g) be a Riemannian manifold and let R denote its Riemann curvature tensor and T_pM the tangent space at a point p of M. (M,g) is said to be *curvature homogeneous* [11] if for every pair of points p, q of M, R_p and R_q are algebraically the same, that is, there exists a linear isometry F from T_pM to T_qM that preserves the curvature tensor: $F^*(R_q) = R_p$. Obviously, every homogeneous Riemannian manifold $(\tilde{M},\tilde{g})$ (i.e. such that the full group of isometries acts transitively on $(\tilde{M},\tilde{g})$) is curvature homogeneous. Further, a homogeneous Riemannian manifold $(\tilde{M},\tilde{g})$ is called a *model space* of a Riemannian manifold (M,g) if, for a fixed point $o \in \tilde{M}$ and for each point $p \in M$, there is a linear isometry F from T_pM to $T_o\tilde{M}$ such that $F^*(\tilde{R}_o) = R_p$. The Riemannian manifolds with a homogeneous model space constitute a proper subclass of the class of all curvature homogeneous spaces (see [7,4]).

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SSDI 0926-2245(94)00005-B

Finally, (M,g) is said to be *semi-symmetric* [10,12] if R satisfies $R_{XY} \cdot R = 0$ for all vector fields X,Y on M. (Here the linear endomorphism R_{XY} acts as a derivation on R.) This means that, at each point p of M, R_p is the same as the curvature tensor of a symmetric space (see e.g. [1]). This symmetric space can change with the point.

The study of curvature homogeneous manifolds was started by I.M. Singer in [11]. He asked whether such spaces are necessarily (locally) homogeneous. K. Sekigawa [8,9] and H. Takagi [14] gave a negative answer to this question by constructing three- and four-dimensional counterexamples. Moreover, these examples have the same curvature tensor as the symmetric spaces $H^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}^2$ and, hence, are in particular semi-symmetric. (H^2 denotes the two-dimensional hyperbolic plane.)

Semi-symmetric curvature homogeneous spaces are those which have at each point the same curvature tensor as a *fixed* symmetric space, i.e., those with a symmetric model space. This class of manifolds has been studied intensively by the last two authors and F. Tricerri [15,16,5,6]. In [15] it is proved that a curvature homogeneous space with irreducible symmetric model is itself locally symmetric and hence locally isometric to that model. Further, in [16], this result is extended to the class where the model is symmetric but without Euclidean factor. Finally, the most general result is obtained in [6] by specializing the local classification theorem for semi-symmetric spaces given by Z.I. Szabó in [12] to the curvature homogeneous case. There the following local structure theorem is proved [6, Theorem 10.1 and Corollary 10.1]:

Let (M,g) be a curvature homogeneous Riemannian manifold with a symmetric model. Then (M,g) is locally isometric to a Riemannian product

$$(M_s, g_s) \times (F_1, h_1) \times \cdots \times (F_r, h_r),$$

where (M_s, g_s) is a symmetric space and (F_i, h_i), $i = 1, \ldots, r$, are locally irreducible Riemannian spaces which are foliated by totally geodesic Euclidean leaves of codimension two and have constant scalar curvature. Each (F_i, h_i) has a symmetric model of the form $S^2 \times \mathbb{R}^{k_i}$ or $H^2 \times \mathbb{R}^{k_i}$, $k_i \geqslant 1$.

Moreover, the same authors also generalized Sekigawa's three-dimensional example and extended it to arbitrary higher dimensions, thus providing examples of the foliated spaces mentioned in the above theorem.

In [3] the second author proves that all *three-dimensional* locally non-homogeneous spaces with a symmetric model are given by the generalized examples of Sekigawa. He obtains this as a side result of his *explicit* local classification of non-symmetric three-dimensional semi-symmetric spaces (i.e., of all Riemannian 3-manifolds whose curvature tensor has constant index of nullity equal to 1).

In this paper we will prove the higher-dimensional analogue. We consider the class of non-symmetric, locally irreducible semi-symmetric spaces (M^{n+2}, g) which are also curvature homogeneous. Each of these spaces is foliated by n-dimensional Euclidean leaves and has the symmetric model space $S^2(\lambda^2) \times \mathbb{R}^n$ or $H^2(-\lambda^2) \times \mathbb{R}^n$, according to whether the scalar curvature is positive or negative. ($S^2(\lambda^2)$ and $H^2(-\lambda^2)$ denote the standard 2-sphere and the hyperbolic plane with constant Gaussian curvature λ^2

or $-\lambda^2$, respectively.) Our main result is that, generically, the only possible spaces are exactly those given by the authors of [6]. Moreover, we will extend this result to the case where we suppose only that the scalar curvature is constant along the leaves instead of being a global constant. The classification we obtain in this way yields a complete classification of all parabolically foliated semi-symmetric spaces in the sense of Z.I. Szabó [13].

The paper is organized as follows. In Section 2 we give a special local form for the metrics and determine the basic system of partial differential equations for our problem. Then, in Section 3, we determine some first integrals of the partial differential equations and distinguish two algebraic cases. We treat these two cases separately in Section 4 and we solve the basic system of partial differential equations to get explicit formulas for the germs of the metrics. This leads to the result mentioned above. Finally, the extension is treated in Section 5.

2. A canonical form for the metrics and the basic system of partial differential equations

Throughout the paper all manifolds, maps, vector fields and differential forms are assumed to be C^∞. We will denote by (M^{n+2}, g) an $(n+2)$-dimensional Riemannian manifold with metric g, by D its Levi Civita connection and by R its Riemann curvature tensor.

In the introduction we mentioned the classification of semi-symmetric spaces by Z.I. Szabó [12]. The most general family of (locally irreducible) semi-symmetric spaces is given by $(n+2)$-dimensional Riemannian manifolds foliated by n-dimensional totally geodesic Euclidean leaves. These are Riemannian spaces (M^{n+2}, g) whose index of nullity $\nu(p)$ is constant along M and equal to n. This means that every tangent space T_pM can be decomposed in the form

$$T_pM = V_p^{(0)} + V_p^{(1)},$$

where $\dim V_p^{(0)} = n$, $\dim V_p^{(1)} = 2$ for all $p \in M$, and $V_p^{(0)}$ is the *null-space* of the Riemannian curvature tensor R_p, i.e.,

$$V_p^{(0)} = \{X \in T_pM \mid R_p(X,Y) = 0 \quad \text{for all } Y \in T_pM\}.$$

Hence the curvature tensor R_p at each point $p \in M$ is the same as that of the space $M' = S^2(\lambda^2) \times \mathbb{R}^n$, or $M' = H^2(-\lambda^2) \times \mathbb{R}^n$, where the sectional curvature $\pm\lambda^2$ depends on the point p, in general. Further, Z.I. Szabó shows that the n-dimensional distribution $V^{(0)}$ on M is completely integrable, and that the integral manifolds of $V^{(0)}$ are totally geodesic and locally Euclidean. This is why M is said to be foliated by n-dimensional Euclidean leaves.

The family of foliated spaces is not described explicitly in Szabó's work. In [3], the second author treated the three-dimensional case and gave the explicit expressions for the germs of metrics by solving a complicated system of partial differential equations. For this purpose he used a special system of local coordinates.

Now, we shall generalize this for arbitrary n and we start by introducing the corresponding local coordinate system. Then we determine the corresponding system of partial differential equations expressing the semi-symmetricity and the curvature homogeneity.

Proposition 2.1. *Let (M^{n+2},g) be an $(n+2)$-dimensional smooth foliated semi-symmetric space. Then, in a neighbourhood U of each point $p \in M$, there are local coordinates $x,w,y^1,\ldots,y^n$ such that $g = \sum_{i=1}^{n+2}(\omega^i)^2$, where*

$$\begin{aligned}
&\omega^1 = f(x,w,y^1,\ldots,y^n)\,dw,\\
&\omega^2 = A(x,w,y^1,\ldots,y^n)\,dx + C(x,w,y^1,\ldots,y^n)\,dw,\\
&\omega^{\alpha+2} = dy^\alpha + H^\alpha(x,w,y^1,\ldots,y^n)\,dw + G^\alpha(x,w,y^1,\ldots,y^n)\,dx,\\
&\alpha = 1,\ldots,n.
\end{aligned} \tag{2.1}$$

Here $fA \neq 0$ and the equations $\omega^1 = \omega^2 = 0$ (or equivalently, $x =$ constant and $w =$ constant) determine the Euclidean leaves of dimension n. $(y^1,\ldots,y^n)$ gives a local coordinate system on each leaf.

Proof. Choose a point p of M and a neighbourhood U' of p. Let $S : D^2 \to U'$ be a surface through p which is transversal to the Euclidean leaves at all cross-points. Then there exists a normal neighbourhood U of p, $U \subset U'$, with the property that each point q of U is projected to exactly one point,$\pi(q)$, of S via some Euclidean leaf. We take any local coordinates (x,w) on S. We can then take a local coordinate system $(x,w,y^1,\ldots,y^n)$ on U such that $w(q)$ and $x(q)$ are defined by $w(\pi(q))$ and $x(\pi(q))$ for each q in U and such that on every leaf of the foliation the coordinates $(y^1,\ldots,y^n)$ are Euclidean coordinates. This last condition means that $E_{\alpha+2} = \partial/\partial y^\alpha$, $\alpha = 1,\ldots,n$, are orthonormal vector fields on U. We extend these to an orthonormal frame field $(E_1,E_2,E_3,\ldots,E_{n+2})$ on a neighbourhood of p (cf. [12]). Let $(\bar\omega^1,\bar\omega^2,\omega^3,\ldots,\omega^{n+2})$ be its dual coframe. In terms of the coordinates $(x,w,y^1,\ldots,y^n)$ we must have

$$\begin{aligned}
&\bar\omega^1 = P_1(x,w,y^1,\ldots,y^n)\,dw + Q_1(x,w,y^1,\ldots,y^n)\,dx,\\
&\bar\omega^2 = P_2(x,w,y^1,\ldots,y^n)\,dw + Q_2(x,w,y^1,\ldots,y^n)\,dx,\\
&\omega^{\alpha+2} = dy^\alpha + H^\alpha(x,w,y^1,\ldots,y^n)\,dw + G^\alpha(x,w,y^1,\ldots,y^n)\,dx,\\
&\alpha = 1,\ldots,n.
\end{aligned}$$

Taking suitable orthogonal linear combinations of $\bar\omega^1$ and $\bar\omega^2$ we can find orthonormal one-forms ω^1 and ω^2 of the form

$$\begin{aligned}
&\omega^1 = f(x,w,y^1,\ldots,y^n)\,dw,\\
&\omega^2 = A(x,w,y^1,\ldots,y^n)\,dx + C(x,w,y^1,\ldots,y^n)\,dw,
\end{aligned}$$

such that $(\omega^1,\omega^2,\omega^3,\ldots,\omega^{n+2})$ is an orthonormal coframe, that is, the metric g is given by

$$g = \sum_{i=1}^{n+2} \omega^i \otimes \omega^i.$$

Because $(\omega^1, \dots, \omega^{n+2})$ is a coframe, we see that $Af \neq 0$. This proves the proposition.

From now on we assume, in addition, that the foliated space (M^{n+2}, g) is curvature homogeneous. So, its curvature is of the type $H^2 \times \mathbb{R}^n$ or $S^2 \times \mathbb{R}^n$, where the constant Gaussian curvature of H^2, or that of S^2, will now be denoted by k.

Consider the components ω_2^1, $\omega_{\alpha+2}^1$, $\omega_{\alpha+2}^2$, $\omega_{\beta+2}^{\alpha+2}$ of the connection form with respect to the coframe $(\omega^1, \dots, \omega^{n+2})$. (Here and in the sequel the Greek indices refer to the y-coordinates and range from 1 to n.) The components satisfy the standard equations (see [2])

$$\begin{aligned}
&d\omega^1 + \omega_2^1 \wedge \omega^2 + \sum_\alpha \omega_{\alpha+2}^1 \wedge \omega^{\alpha+2} = 0,\\
&d\omega^2 + \omega_1^2 \wedge \omega^1 + \sum_\alpha \omega_{\alpha+2}^2 \wedge \omega^{\alpha+2} = 0,\\
&d\omega^{\alpha+2} + \omega_1^{\alpha+2} \wedge \omega^1 + \omega_2^{\alpha+2} \wedge \omega^2 + \sum_\beta \omega_{\beta+2}^{\alpha+2} \wedge \omega^{\beta+2} = 0,\\
&\omega_2^1 + \omega_1^2 = 0, \quad \omega_{\alpha+2}^1 + \omega_1^{\alpha+2} = 0, \omega_{\alpha+2}^2 + \omega_2^{\alpha+2} = 0, \quad \omega_{\beta+2}^{\alpha+2} + \omega_{\alpha+2}^{\beta+2} = 0.
\end{aligned} \tag{2.2}$$

Because the curvature tensor of M is the same as that of $H^2 \times \mathbb{R}^n$ or $S^2 \times \mathbb{R}^n$, the components Ω_2^1, $\Omega_{\alpha+2}^1$, $\Omega_{\alpha+2}^2$, and $\Omega_{\beta+2}^{\alpha+2}$ of the curvature form (with respect to the coframe $(\omega^1, \omega^2, \omega^{\alpha+2})$) must satisfy

$$\begin{aligned}
&\Omega_2^1 = -\Omega_1^2 = k\,\omega^1 \wedge \omega^2,\\
&\Omega_{\alpha+2}^1 = \Omega_{\alpha+2}^2 = \Omega_{\beta+2}^{\alpha+2} = 0,
\end{aligned}$$

where k is a non-zero constant. By the standard formulas, this is equivalent to the following system of equations for the components of the connection form:

$$d\omega_2^1 + \sum_\alpha \omega_{\alpha+2}^1 \wedge \omega_2^{\alpha+2} = k\,\omega^1 \wedge \omega^2, \tag{2.3}$$

$$d\omega_{\alpha+2}^1 + \omega_2^1 \wedge \omega_{\alpha+2}^2 + \sum_\beta \omega_{\beta+2}^1 \wedge \omega_{\alpha+2}^{\beta+2} = 0, \quad \alpha = 1, \dots, n, \tag{2.4}$$

$$d\omega_{\alpha+2}^2 + \omega_1^2 \wedge \omega_{\alpha+2}^1 + \sum_\beta \omega_{\beta+2}^2 \wedge \omega_{\alpha+2}^{\beta+2} = 0, \quad \alpha = 1, \dots, n, \tag{2.5}$$

$$d\omega_{\beta+2}^{\alpha+2} + \omega_1^{\alpha+2} \wedge \omega_{\beta+2}^1 + \omega_2^{\alpha+2} \wedge \omega_{\beta+2}^2 + \sum_\gamma \omega_{\gamma+2}^{\alpha+2} \wedge \omega_{\beta+2}^{\gamma+2} = 0, \tag{2.6}$$

$\alpha, \beta = 1, \dots, n.$

Taking the exterior derivatives of (2.3)–(2.6) and then substituting properly from (2.3)–(2.6) into the derived equations, we obtain

$$d(\omega^1 \wedge \omega^2) = 0, \tag{2.7}$$

$$\omega^1 \wedge \omega^2 \wedge \omega^2_{\alpha+2} = 0, \quad \alpha = 1, \dots, n, \tag{2.8}$$

$$\omega^1 \wedge \omega^2 \wedge \omega^1_{\alpha+2} = 0, \quad \alpha = 1, \dots, n. \tag{2.9}$$

The condition (2.7) is easily reduced (using (2.1)) to

$$\frac{\partial}{\partial y^\alpha}(Af) = 0, \quad \alpha = 1, \dots, n$$

or

$$Af = M(x, w) \neq 0. \tag{2.10}$$

(This is in fact equivalent to the condition that the sectional curvature k from (2.3) does not depend on y^α.)

Next we write

$$\begin{aligned}
\omega^1_2 &= a^1_{21}\omega^1 + a^1_{22}\omega^2 + \sum_\alpha a^1_{2\alpha}\omega^{\alpha+2}, \\
\omega^1_{\alpha+2} &= b^1_{\alpha 1}\omega^1 + b^1_{\alpha 2}\omega^2 + \sum_\beta b^1_{\alpha\beta}\omega^{\beta+2}, \quad \alpha = 1, \dots, n, \\
\omega^2_{\alpha+2} &= c^2_{\alpha 1}\omega^1 + c^2_{\alpha 2}\omega^2 + \sum_\beta c^2_{\alpha\beta}\omega^{\beta+2}, \quad \alpha = 1, \dots, n, \\
\omega^{\alpha+2}_{\beta+2} &= d^\alpha_{\beta 1}\omega^1 + d^\alpha_{\beta 2}\omega^2 + \sum_\gamma d^\alpha_{\beta\gamma}\omega^{\gamma+2}, \quad \alpha, \beta = 1, \dots, n,
\end{aligned} \tag{2.11}$$

and we calculate the a^i_{jk}, b^i_{jk}, c^i_{jk} and d^i_{jk} from the equations (2.2). First, (2.8) and (2.9) imply that

$$b^1_{\alpha\beta} = c^2_{\alpha\beta} = 0, \quad \alpha, \beta = 1, \dots, n. \tag{2.12}$$

After a routine calculation we obtain the following expressions for the connection forms:

$$\begin{aligned}
\omega^1_2 &= \psi\Big(f'_x - \sum_\alpha f'_\alpha G^\alpha\Big)\,\omega^1 - B\,\omega^2 + \sum_\alpha B_\alpha\,\omega^{\alpha+2}, \\
\omega^1_{\alpha+2} &= \psi A f'_\alpha\,\omega^1 + B_\alpha\,\omega^2, \\
\omega^2_{\alpha+2} &= \big(\psi(AC'_\alpha - CA'_\alpha) - B_\alpha\big)\,\omega^1 + \psi f A'_\alpha\,\omega^2, \\
\omega^{\alpha+2}_{\beta+2} &= \psi\big(A(H^\alpha)'_\beta - C(G^\alpha)'_\beta\big)\,\omega^1 + \psi f (G^\alpha)'_\beta\,\omega^2,
\end{aligned} \tag{2.13}$$

where

$$\psi = (Af)^{-1} = M^{-1}, \tag{2.14}$$

$$B_\alpha = a^1_{2\alpha} = \frac{1}{2}\psi\Big((H^\alpha)'_x - (G^\alpha)'_w + (AC'_\alpha - CA'_\alpha) \\ + \sum_{\beta\neq\alpha}\big(H^\beta(G^\alpha)'_\beta - G^\beta(H^\alpha)'_\beta\big)\Big), \tag{2.15}$$

$$B = a^2_{12} = \psi\Big(A'_w - C'_x + \sum_\alpha\big(C'_\alpha G^\alpha - A'_\alpha H^\alpha\big)\Big). \tag{2.16}$$

Moreover, (2.12) implies

$$(H^\alpha)'_\alpha = 0, \tag{2.17}$$

$$(G^\alpha)'_\alpha = 0. \tag{2.18}$$

In terms of dw, dx and dy^α the connection forms are expressed as follows:

$$\begin{aligned} \omega^1_2 &= \Big(-AB + \sum_\alpha B_\alpha G^\alpha\Big)\,dx + R\,dw + \sum_\alpha B_\alpha\,dy^\alpha, \\ \omega^1_{\alpha+2} &= AB_\alpha\,dx + S_\alpha\,dw, \\ \omega^2_{\alpha+2} &= A'_\alpha\,dx + T_\alpha\,dw, \\ \omega^{\alpha+2}_{\beta+2} &= (G^\alpha)'_\beta\,dx + (H^\alpha)'_\beta\,dw, \end{aligned} \tag{2.19}$$

where

$$R = \psi f f'_x - CB + \sum_\alpha B_\alpha H^\alpha - f\psi\sum_\alpha f'_\alpha G^\alpha, \tag{2.20}$$

$$S_\alpha = f'_\alpha + CB_\alpha, \tag{2.21}$$

$$T_\alpha = C'_\alpha - fB_\alpha. \tag{2.22}$$

In this notation the curvature conditions (2.3)–(2.6) have the following form (where $\alpha,\beta,\gamma = 1,\dots,n$):

$$\frac{\partial(AB)}{\partial y^\alpha} - \sum_\beta \frac{\partial(B_\beta G^\beta)}{\partial y^\alpha} + \frac{\partial B_\alpha}{\partial x} = 0, \tag{A1}$$

$$\frac{\partial B_\alpha}{\partial w} - \frac{\partial R}{\partial y^\alpha} = 0, \tag{A2}$$

$$\frac{\partial(AB)}{\partial w} - \sum_\alpha \frac{\partial(B_\alpha G^\alpha)}{\partial w} + \frac{\partial R}{\partial x} - \sum_\alpha(AB_\alpha T_\alpha - A'_\alpha S_\alpha) = -kAf, \tag{A3}$$

$$\frac{\partial(AB_\alpha)}{\partial y^\beta} + A'_\alpha B_\beta = 0, \tag{B1}$$

$$\frac{\partial S_\alpha}{\partial y^\beta} + T_\alpha B_\beta = 0, \tag{B2}$$

$$\frac{\partial S_\alpha}{\partial x} - (ABT_\alpha + RA'_\alpha) - \frac{\partial (AB_\alpha)}{\partial w} + T_\alpha \sum_\beta B_\beta G^\beta + \sum_{\beta \neq \alpha} \left(AB_\beta (H^\beta)'_\alpha - S_\beta (G^\beta)'_\alpha\right) = 0, \tag{B3}$$

$$A''_{\alpha\beta} - AB_\alpha B_\beta = 0, \tag{C1}$$

$$\frac{\partial T_\alpha}{\partial y^\beta} - S_\alpha B_\beta = 0, \tag{C2}$$

$$-A''_{\alpha w} + \frac{\partial T_\alpha}{\partial x} + A(BS_\alpha + RB_\alpha) - S_\alpha \sum_\beta B_\beta G^\beta + \sum_{\beta \neq \alpha} \left(A'_\beta (H^\beta)'_\alpha - T_\beta (G^\beta)'_\alpha\right) = 0, \tag{C3}$$

$$(G^\alpha)''_{\beta\gamma} = 0, \tag{D1}$$

$$(H^\alpha)''_{\beta\gamma} = 0, \tag{D2}$$

$$(H^\alpha)''_{\beta x} - (G^\alpha)''_{\beta w} + A(B_\beta S_\alpha - B_\alpha S_\beta) + (A'_\beta T_\alpha - A'_\alpha T_\beta) + \sum_\gamma \left((G^\alpha)'_\gamma (H^\gamma)'_\beta - (G^\gamma)'_\beta (H^\alpha)'_\gamma\right) = 0, \quad \alpha \neq \beta. \tag{D3}$$

3. First integrals of the basic PDE and generic points

In this section we will determine how the functions H^α, G^α, AS_α, AT_α, AC, $f^2 + C^2$ and A^2 depend on the variables $y^1, \ldots, y^n$. We will see that they are polynomials in these variables of degree not greater than two. Moreover, we will find a coordinate transformation such that A^2 will depend on at most four variables, namely w, x, y^1 and y^2.

a) The functions H^α and G^α

From (D1) and (D2) it is immediate that H^α and G^α are linear functions in the y^β. So we can write

$$H^\alpha = \sum_\beta h^\alpha_\beta(x,w)\, y^\beta + h^\alpha_0(x,w), \tag{3.1}$$

$$G^\alpha = \sum_\beta g^\alpha_\beta(x,w)\, y^\beta + g^\alpha_0(x,w). \tag{3.2}$$

We have also

$$h^\alpha_\beta + h^\beta_\alpha = 0, \quad g^\alpha_\beta + g^\beta_\alpha = 0, \tag{3.3}$$

due to the last formula in (2.19) and the fact that $\omega^{\alpha+2}_{\beta+2} + \omega^{\beta+2}_{\alpha+2} = 0$.

b) The functions AS_α and AT_α

Using (B1), (B2), (C1) and (C2) we easily find

$$(AS_\alpha)''_{\beta\gamma} = 0 \quad \text{and } (AT_\alpha)''_{\beta\gamma} = 0.$$

So, we have

$$AS_\alpha = \sum_\beta s_{\alpha\beta}(x, w)\, y^\beta + s_{\alpha 0}(x, w), \tag{3.4}$$

$$AT_\alpha = \sum_\beta t_{\alpha\beta}(x, w)\, y^\beta + t_{\alpha 0}(x, w). \tag{3.5}$$

c) The function AC

Using first (C2) and then the definitions (2.21) and (2.22) of T_α and S_α together with (C1), we derive

$$(AT_\alpha)'_\beta = (A'_\beta C)'_\alpha - A'_\beta f B_\alpha + A f'_\alpha B_\beta. \tag{3.6}$$

On the other hand, using first the definition of T_α and then (B1), we get

$$(AT_\alpha)'_\beta = (AC'_\alpha)'_\beta - A f'_\beta B_\alpha + A'_\alpha f B_\beta. \tag{3.7}$$

Combining (3.6) and (3.7) we obtain

$$(AT_\alpha)'_\beta + (AT_\beta)'_\alpha = (AC)''_{\alpha\beta}. \tag{3.8}$$

This implies in particular that $(AC)'''_{\alpha\beta\gamma} = 0$. So, we get the following expression for AC:

$$AC = \sum_{\alpha,\beta} b_{\alpha\beta}(x, w)\, y^\alpha y^\beta + \sum_\gamma b_\gamma(x, w)\, y^\gamma + b(x, w), \tag{3.9}$$

where $b_{\alpha\beta} = b_{\beta\alpha}$. Moreover, we find from (3.8)

$$t_{\alpha\beta} + t_{\beta\alpha} = 2b_{\alpha\beta}. \tag{3.10}$$

In a similar way we derive

$$(AS_\alpha)'_\beta + (AS_\beta)'_\alpha = (Af)''_{\alpha\beta}. \tag{3.11}$$

As Af is independent of the variables $y^1, \ldots, y^n$, we get

$$s_{\alpha\beta} + s_{\beta\alpha} = 0. \tag{3.12}$$

d) The function $f^2 + C^2$

Using (2.21), (2.22), (B2) and (C2) we obtain

$$\begin{aligned}(f^2 + C^2)'''_{\alpha\beta\gamma} &= 2(fS_\alpha + CT_\alpha)''_{\beta\gamma} \\ &= 2(S_\alpha S_\beta + T_\alpha T_\beta)'_\gamma = 0.\end{aligned}$$

Hence we have the following expression for $f^2 + C^2$:

$$f^2 + C^2 = \sum_{\alpha,\beta} \varphi_{\alpha\beta}(x,w)\, y^\alpha y^\beta + \sum_\gamma \varphi_\gamma(x,w)\, y^\gamma + \varphi(x,w), \tag{3.13}$$

where $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}$.

e) The function A^2

Using (C1) and (B1) we easily calculate

$$(A^2)'''_{\alpha\beta\gamma} = 0$$

and so we have

$$A^2 = \sum_{\alpha,\beta} a_{\alpha\beta}(x,w)\, y^\alpha y^\beta + \sum_\gamma a_\gamma(x,w)\, y^\gamma + a(x,w), \tag{3.14}$$

where $a_{\alpha\beta} = a_{\beta\alpha}$.

Now we will show that it is always possible to find a local coordinate system $(x, w, u^1, \ldots, u^n)$ and an orthonormal coframe $(\omega^1, \omega^2, \tau^3, \ldots, \tau^{n+2})$ of the form (2.1) such that A^2 is expressed in the form

$$A^2 = \lambda_1 (u^1)^2 + \lambda_2 (u^2)^2 + \bar{a}_1 u^1 + \bar{a}_2 u^2 + \bar{a}, \tag{3.15}$$

where λ_1, λ_2, $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}$ are functions of x and w only. To do this, suppose that we have local coordinates $(x, w, y^1, \ldots, y^n)$ and an orthonormal coframe $(\omega^1, \omega^2, \omega^3, \ldots, \omega^{n+2})$ of the form (2.1). Then A^2 has the form (3.14). Using (C1) we can see easily that $2A^2 B_\alpha B_\beta = (A^2)''_{\alpha\beta} - 2A'_\alpha A'_\beta$, and hence the rank of the matrix $\left[(A^2)''_{\alpha\beta} - 2A'_\alpha A'_\beta\right]$ is not greater than one, i.e.,

$$a_{\alpha\alpha} a_{\beta\beta} - (a_{\alpha\beta})^2 - a_{\alpha\alpha}(A'_\beta)^2 - a_{\beta\beta}(A'_\alpha)^2 + 2a_{\alpha\beta} A'_\alpha A'_\beta = 0$$

for all α and β. Multiplying with A^2 we get

$$(a_{\alpha\alpha} a_{\beta\beta} - (a_{\alpha\beta})^2) A^2 - \tfrac{1}{4} a_{\alpha\alpha} \left[(A^2)'_\beta\right]^2 - \tfrac{1}{4} a_{\beta\beta} \left[(A^2)'_\alpha\right]^2 + \tfrac{1}{2} a_{\alpha\beta} (A^2)'_\alpha (A^2)'_\beta = 0.$$

Substituting from (3.14) and taking the coefficient of any $(y^\gamma)^2$, $\gamma \neq \alpha, \gamma \neq \beta$, we get

$$\begin{vmatrix} a_{\alpha\alpha} & a_{\beta\alpha} & a_{\gamma\alpha} \\ a_{\alpha\beta} & a_{\beta\beta} & a_{\gamma\beta} \\ a_{\alpha\gamma} & a_{\beta\gamma} & a_{\gamma\gamma} \end{vmatrix} = 0.$$

Taking the coefficient of y^γ, we obtain

$$\begin{vmatrix} a_{\alpha\alpha} & a_{\beta\alpha} & a_{\gamma\alpha} \\ a_{\alpha\beta} & a_{\beta\beta} & a_{\gamma\beta} \\ a_\alpha & a_\beta & a_\gamma \end{vmatrix} = 0.$$

Hence,

$$\operatorname{rank} \begin{pmatrix} a_{11} & \dots & a_{n\,1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{n\,n} \\ a_1 & \dots & a_n \end{pmatrix} \leqslant 2. \tag{3.16}$$

Using a well-known theorem of linear algebra we can find an orthogonal matrix Q whose entries are functions of x and w only and such that, in the new coordinates $(\tilde{u}^1, \dots, \tilde{u}^n)$ given by

$$\begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^n \end{pmatrix} = Q \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix},$$

A^2 takes on the form

$$A^2 = \lambda_1(\tilde{u}^1)^2 + \lambda_2(\tilde{u}^2)^2 + \sum_{\gamma=1}^{n} \tilde{a}_\gamma\, \tilde{u}^\gamma + \tilde{a}. \tag{3.17}$$

Now, introducing the column matrices

$$\begin{aligned} &Y = (y^\alpha), && \tilde{U} = (\tilde{u}^\alpha), \\ &H = (H^\alpha), && G = (G^\alpha), \\ &\omega = (\omega^{\alpha+2}), && \alpha = 1, \dots, n, \end{aligned}$$

we have

$$\begin{aligned} Q\omega &= Q\,dY + QH\,dw + QG\,dx \\ &= d(QY) - (dQ)Y + QH\,dw + QG\,dx \\ &= dU + (QH - P_2Q^{-1}U)\,dw + (QG - P_1Q^{-1}U)\,dx, \end{aligned}$$

where P_1 and P_2 are matrix functions depending on x and w only and satisfying $dQ = P_1\,dx + P_2\,dw$. If we put $\tau = Q\omega$, then $(\tau^3, \dots, \tau^{n+2})$ is an orthonormal set of one-forms and the coframe $(\omega^1, \omega^2, \tau^3, \dots, \tau^{n+2})$ is of the form (2.1) with respect to the coordinates $(x, w, \tilde{u}^1, \dots, \tilde{u}^n)$. Moreover, A^2 has the form (3.17). In this new coordinate system the conditions (A1)–(D3) still hold. In particular we must have (just as in (3.16))

$$\operatorname{rank} \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \dots & \tilde{a}_n \end{pmatrix} \leqslant 2. \tag{3.18}$$

We consider the three different possibilities.

1) $\lambda_1\lambda_2 \neq 0$. In this case (3.18) implies that $\tilde{a}_3 = \cdots = \tilde{a}_n = 0$ and A^2 is of the form (3.15).

2) $\lambda_1 \neq 0$, $\lambda_2 = 0$ (or $\lambda_1 = 0$, $\lambda_2 \neq 0$). If $\tilde{a}_3 = \cdots = \tilde{a}_n = 0$, A^2 is of the form (3.15). If not, we make another orthogonal coordinate transformation of the form

$$\begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = Q_1(w,x) \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^n \end{pmatrix},$$

where the first two relations read $u^1 = \tilde{u}^1$, $u^2 = \mu(w,x)(\tilde{a}_2\tilde{u}^2 + \cdots + \tilde{a}_n\tilde{u}^n)$, $\mu \neq 0$. By repeating the procedure above (see the text between the formulas (3.17) and (3.18)) we can find easily an orthonormal coframe of the form (2.1) with respect to $(x, w, u^1, \dots, u^n)$, where A^2 is of the form (3.15) (with $\lambda_2 = 0$).

3) $\lambda_1 = \lambda_2 = 0$. If $\tilde{a}_3 = \cdots = \tilde{a}_n = 0$, A^2 is of the form (3.15). If not, we make again an orthogonal coordinate transformation

$$\begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = Q_2(w,x) \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^n \end{pmatrix},$$

which makes $u^1 = 0$ equivalent to $\tilde{a}_1\tilde{u}^1 + \cdots + \tilde{a}_n\tilde{u}^n = 0$. Again we can find an orthonormal coframe of the form (2.1) with respect to the coordinates $(x, w, u^1, \dots, u^n)$, where A^2 is of the form (3.15) (with $\lambda_1 = \lambda_2 = 0$).

In the sequel we will always work in a local coordinate system $(x, w, u^1, \dots, u^n)$ such that A^2 has the simple form (3.15). We derive an additional formula for λ_1, λ_2, $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}$ to be used later. For this purpose, we write down the expressions for the B_α. Starting from (C1) and using (3.15) we obtain

$$\begin{aligned} &B_\alpha = 0 \quad \text{for } \alpha \geqslant 3, \\ &(B_1)^2 = \big(\lambda_1\lambda_2(u^2)^2 + \lambda_1\bar{a}_2u^2 + \big(\lambda_1\bar{a} - (\bar{a}_1)^2/4\big)\big)/A^4, \\ &(B_2)^2 = \big(\lambda_1\lambda_2(u^1)^2 + \lambda_2\bar{a}_1u^1 + \big(\lambda_2\bar{a} - (\bar{a}_2)^2/4\big)\big)/A^4, \\ &B_1B_2 = -\big(\lambda_1\lambda_2u^1u^2 - (\lambda_1\bar{a}_2/2)\,u^1 - (\lambda_2\bar{a}_1/2)\,u^2 - \bar{a}_1\bar{a}_2/4\big)/A^4. \end{aligned} \tag{3.19}$$

Expressing the obvious equality $(B_1)^2(B_2)^2 = (B_1B_2)^2$ by means of (3.19), we find the wanted condition

$$4\lambda_1\lambda_2\bar{a} - \lambda_1(\bar{a}_2)^2 - \lambda_2(\bar{a}_1)^2 = 0. \tag{3.20}$$

Proposition 3.1. *At any basic point $p \in M$ of the local coordinates we have $\lambda_1\lambda_2 = 0$.*

Proof. Suppose that, to the contrary, $\lambda_1\lambda_2 \neq 0$ at p. We write (3.15) in the form $A^2 = \lambda_1\big(u^1 + \bar{a}_1/(2\lambda_1)\big)^2 + \lambda_2\big(u^2 + \bar{a}_2/(2\lambda_2)\big)^2 + \big(\bar{a} - (\bar{a}_1)^2/(4\lambda_1) - (\bar{a}_2)^2/(4\lambda_2)\big)$. Because of (3.20) the last term vanishes. If we define new coordinates

$$\begin{aligned} v^1 &= u^1 + \bar{a}_1/(2\lambda_1), \\ v^2 &= u^2 + \bar{a}_2/(2\lambda_2), \\ v^\alpha &= u^\alpha \quad (\alpha \geqslant 3), \end{aligned}$$

we get the following form for A^2 in a neighbourhood of p:

$$A^2 = \lambda_1(v^1)^2 + \lambda_2(v^2)^2. \tag{3.21}$$

Now, we get the following analogues of the formulas (3.9) and (3.13):

$$\begin{aligned} AC &= \sum_{\alpha,\beta} b_{\alpha\beta}(x,w)\, v^\alpha v^\beta + \sum_\gamma b_\gamma(x,w)\, v^\gamma + b(x,w), \\ f^2 + C^2 &= \sum_{\alpha,\beta} \varphi_{\alpha\beta}(x,w)\, v^\alpha v^\beta + \sum_\gamma \varphi_\gamma(x,w)\, v^\gamma + \varphi(x,w). \end{aligned}$$

If we express the equality $A^2(f^2 + C^2) = (Af)^2 + (AC)^2$ using (2.10), (3.21) and the above formulas and compare the terms which are independent of the v^α, we find

$$M^2 + b^2 = 0$$

and hence $Af = M = 0$, which is a contradiction.

In the sequel we can always assume that $\lambda_2 = 0$ at p. (The case $\lambda_1 = 0$ is reduced to the first one by a renumeration of the coordinates.) We shall call $p \in M$ a *generic point* if either I) $\lambda_1 = \lambda_2 = 0$ in a neighbourhood of p, or II) $\lambda_1 \neq 0$ at p, i.e., $\lambda_2 = 0$ in a neighbourhood of p. We can see easily that the generic points of M form a dense open subset (under the convention made about the numeration of λ_1 and λ_2). The non-generic points are those for which $\lambda_1(p) = \lambda_2(p) = 0$ but λ_2 is not identically zero in a neighbourhood of p.

Next we have

Proposition 3.2. *In a neighbourhood of any generic point $p \in M$, the function A^2 is a polynomial of the form*

$$A^2 = \lambda_1(u^1)^2 + \bar{a}_1 u^1 + \bar{a} \tag{3.22}$$

with respect to a local coordinate system $(x, w, u^1, \ldots, u^n)$ and to an orthonormal coframe $(\omega^1, \ldots, \omega^{n+2})$ as in (2.1).

Proof. If $\lambda_1(p) \neq 0$ then (3.20) implies $\bar{a}_2 = 0$ and from (3.15) we get (3.22). If $\lambda_1 = \lambda_2 = 0$ in a neighbourhood of p, then we have, according to (3.19), $\bar{a}_1 = B_1 = 0$ and $\bar{a}_2 = B_2 = 0$. Hence $A^2 = \bar{a}$. This concludes the proof.

Further, we also have the following expressions for the functions B_α:

$$A^2 B_1 = a_0, \quad B_\alpha = 0 \quad (\alpha \geqslant 2), \tag{3.23}$$

where

$$4(a_0)^2 - 4\bar{a}\lambda_1 + (\bar{a}_1)^2 = 0. \tag{3.24}$$

4. The germs of metrics solving the basic PDE system

In this section we consider the two above types of the generic points and we are going to prove first that the function A depends only on x and w in a neighbourhood of any "regular" point (to be defined later). Then we are going to find an explicit formula for our Riemannian metrics in the neighbourhoods of all regular points.

I. *First case*: $\lambda_1 = \lambda_2 = 0$ *in a neighbourhood of* p. From the proof of Proposition 3.2. we see that A has the simple form $A = c(x, w)$ on a neighbourhood of p.

II. *Second case*: $A^2 = \lambda_1(u^1)^2 + \bar{a}_1 u^1 + \bar{a}$, $\lambda_1 \neq 0$, *on a neighbourhood of* p. Again we express the equality $A^2(f^2 + C^2) = (Af)^2 + (AC)^2$, this time using (2.10), (3.9), (3.13) and (3.22). Comparing the coefficients of the corresponding polynomials in $u^1, \ldots, u^n$, we get the following list of relations:

$$\begin{array}{ll}
(1) & b_{\alpha\beta} = 0, \quad \alpha, \beta \geqslant 2, \\
(2) & \lambda_1 \varphi_{11} = (b_{11})^2, \\
(3) & \lambda_1 \varphi_{1\beta} = 2b_{11}b_{1\beta}, \quad \beta \geqslant 2, \\
(4) & \lambda_1 \varphi_{\beta\beta} = 4(b_{1\beta})^2, \quad \beta \geqslant 2, \\
(5) & \lambda_1 \varphi_1 + \bar{a}_1 \varphi_{11} = 2b_{11}b_1, \\
(6) & \lambda_1 \varphi_{\beta\gamma} = 4b_{1\beta}b_{1\gamma}, \quad \beta \neq 1 \neq \gamma \neq \beta, \\
(7) & \lambda_1 \varphi_\beta + 2\bar{a}_1 \varphi_{1\beta} = 2b_{11}b_\beta + 4b_{1\beta}b_1, \quad \beta \geqslant 2, \\
(8) & \bar{a}_1 \varphi_{\beta\beta} = 4b_{1\beta}b_\beta, \quad \beta \geqslant 2, \\
(9) & \bar{a}_1 \varphi_{\beta\gamma} = 2b_{1\beta}b_\gamma + 2b_{1\gamma}b_\beta, \quad \beta \neq 1 \neq \gamma \neq \beta, \\
(10) & \bar{a}\varphi_{\beta\beta} = (b_\beta)^2, \quad \beta \geqslant 2, \\
(11) & \lambda_1 \varphi + \bar{a}_1 \varphi_1 + \bar{a}\varphi_{11} = 2b_{11}b + (b_1)^2, \\
(12) & \bar{a}\varphi_{\beta\gamma} = b_\beta b_\gamma, \quad \beta \neq 1 \neq \gamma \neq \beta, \\
(13) & \bar{a}_1 \varphi_\beta + 2\bar{a}\varphi_{1\beta} = 4b_{1\beta}b + 2b_1 b_\beta, \quad \beta \geqslant 2, \\
(14) & \bar{a}_1 \varphi + \bar{a}\varphi_1 = 2b_1 b, \\
(15) & \bar{a}\varphi_\beta = 2b_\beta b, \quad \beta \geqslant 2, \\
(16) & \bar{a}\varphi = M^2 + b^2.
\end{array} \tag{4.1}$$

Because $M \neq 0$, (4.1)(16) implies that

$$\bar{a} \neq 0. \tag{4.2}$$

We shall now distinguish two subcases which do not necessarily cover all the generic points.

A) *First subcase*: $\varphi_{\beta\beta} \neq 0$ *for some* $\beta \geqslant 2$ *at the point* p. Using (4.1)(4), (8) and (10), we easily calculate

$$((\bar{a}_1)^2 - 4\lambda_1\bar{a})(\varphi_{\beta\beta})^2 = 0.$$

From (3.23) and (3.24) we see that $B_1 = 0$. (C1) then says that A is a linear function of the variable u^1 and (B2), together with (2.21), implies that the same holds for f. We know, however, from (2.10) that Af does not depend on u^1. So, both A and f are independent of u^1 and hence $\lambda_1 = \bar{a}_1 = 0$. Consequently, A has the simple form $A = c(x, w)$ on a neighbourhood of p.

B) *Second subcase*: $\varphi_{\beta\beta} = 0$ *for all* $\beta \geqslant 2$ *on a neighbourhood of* p. In this case, it follows immediately from the equations (4.1) that we also have

$$b_{1\beta} = b_\beta = \varphi_\beta = \varphi_{1\beta} = \varphi_{\beta\gamma} = 0,$$

for all $\beta, \gamma \geqslant 2$. Hence, we obtain

$$\begin{aligned}
&A^2 = \lambda_1(u^1)^2 + \bar{a}_1 u^1 + \bar{a},\\
&Af = M,\\
&AC = b_{11}(u^1)^2 + b_1 u^1 + b,\\
&f^2 + C^2 = \varphi_{11}(u^1)^2 + \varphi_1 u^1 + \varphi.
\end{aligned}$$

Essentially, we have a three-dimensional situation here. We now refer to [3], where it is shown, using the notion of an asymptotic foliation, that, with the expressions for the functions A^2, Af, AC and f^2+C^2 given above, there exists a coordinate transformation for the variables x and w annihilating the function a_0 given by (3.23). As in the previous case, we can first conclude that $B_1 = 0$. Then in the new coordinates, A and f are linear functions of u^1 and hence again $A = c(x, w)$ on a neighbourhood of p.

Now, a generic point $p \in M$ is said to be *regular* if either the case I, or one of the subcases A) or B) of case II occurs. Obviously, the regular points form an open dense subset of the set of all generic points and hence an open dense subset of M.

In the sequel we suppose that $p \in M$ is a regular point, i.e.,

$$A = c(x, w) \tag{4.3}$$

holds in a neighbourhood of p. We again express the equality $A^2(f^2 + C^2) = (Af)^2 + (AC)^2$, using now (2.10), (3.9), (3.13) and (4.3). We find

$$\begin{aligned}
&b_{\alpha\beta} = 0, \quad \alpha, \beta = 1, \ldots, n,\\
&c^2\varphi_{\alpha\beta} = b_\alpha b_\beta, \quad \alpha, \beta = 1, \ldots, n,\\
&c^2\varphi_\alpha = 2b_\alpha b, \quad \alpha = 1, \ldots, n,\\
&c^2\varphi = M^2 + b^2.
\end{aligned} \tag{4.4}$$

So, we have

$$\begin{aligned} &A = c, \\ &f = (M/c), \\ &C = \sum_{\alpha}(b_\alpha/c)\, u^\alpha + (b/c), \\ &H^\alpha = \sum_{\beta} h^\alpha_\beta\, u^\beta + h^\alpha_0, \quad h^\alpha_\beta + h^\beta_\alpha = 0, \\ &G^\alpha = \sum_{\beta} g^\alpha_\beta\, u^\beta + g^\alpha_0, \quad g^\alpha_\beta + g^\beta_\alpha = 0. \end{aligned} \tag{4.5}$$

Further we recall that $B_\alpha = 0$ for all α. From (4.5) and the defining formulas (2.21) and (2.22) we can calculate also S_α and T_α. We obtain

$$\begin{aligned} &B_\alpha = 0, && \alpha = 1, \dots, n, \\ &S_\alpha = 0, && \alpha = 1, \dots, n, \\ &T_\alpha = b_\alpha/c, && \alpha = 1, \dots, n. \end{aligned} \tag{4.6}$$

We will now find a coordinate transformation after which the functions G^α disappear and the form of A, f and C will not change. We use the matrix notation in the sequel. Put

$$\begin{aligned} &G = (g^\alpha_\beta), \quad H = (h^\alpha_\beta), \quad G_0 = (g^\alpha_0), \quad H_0 = (h^\alpha_0), \\ &\underline{u} = (u^\alpha), \quad \omega = (\omega^{\alpha+2}), \quad \alpha = 1, \dots, n, \end{aligned}$$

where $\omega^{\alpha+2} = du^\alpha + G^\alpha\, dx + H^\alpha\, dw$. G_0, H_0, $\underline{v}$ and ω are column vectors. Consider the following differential equation for a matrix function $\phi(x, w)$ with $n \times n$ entries:

$$\frac{d\phi}{dx} = \phi G. \tag{4.7}$$

Because G is skew-symmetric, we see easily that for every solution ϕ of (4.7), $\phi({}^t\phi)$ is independent of x. If we choose the initial value $\phi(0, w)$ as an orthogonal matrix function, then the corresponding solution $\phi(x, w)$ of (4.7) is an orthogonal matrix function, too. Further put

$$\underline{\lambda} = \int_{x_0}^{x} \phi G_0\, dx. \tag{4.8}$$

Then $\underline{\lambda}$ is a column matrix function depending only on x and w. Introduce new variables $v^1, \dots, v^n$ instead of $u^1, \dots, u^n$ by the formula

$$\underline{v} = \phi \underline{u} + \underline{\lambda}. \tag{4.9}$$

Then

$$d\underline{v} = (d\phi)\underline{u} + \phi(d\underline{u}) + d\underline{\lambda}, \tag{4.10}$$

with

$$d\phi = \phi(G\, dx + \underline{p}\, dw), \quad d\underline{\lambda} = \phi(G_0\, dx + \underline{p}_0\, dw), \tag{4.11}$$

where $\underline{p}$, $\underline{p}_0$ are new matrix functions depending only on x and w. ($\underline{p}$ is skew-symmetric and $\underline{p}_0$ is a column matrix function.) From (4.10) and (4.11) we get

$$d\underline{v} = \phi\big((G\underline{u} + G_0)\,dx + (\underline{p}\underline{u} + \underline{p}_0)\,dw\big) + \phi\,d\underline{u}. \tag{4.12}$$

On the other hand, the relations for ω^α (with G^α and H^α given by (4.5)) have the matrix form

$$\omega = d\underline{u} + (G\underline{u} + G_0)\,dx + (H\underline{u} + H_0)\,dw.$$

So, we obtain

$$\phi\omega = d\underline{v} + \{\phi(H - \underline{p})\phi^{-1}(\underline{v} - \underline{\lambda}) + \phi(H_0 - \underline{p}_0)\}\,dw. \tag{4.13}$$

Putting $\underline{\tau} = \phi\omega$, we have obtained a new orthonormal coframe $\omega^1, \omega^2, \tau^3, \dots, \tau^{n+2}$, such that

$$\tau^{\alpha+2} = dv^\alpha + \Big(\sum_\beta \bar{h}^\alpha_\beta v^\beta + \bar{h}^\alpha_0\Big)\,dw, \tag{4.14}$$

where $\bar{h}^\alpha_\beta$ and $\bar{h}^\alpha_0$ depend only on x and w. With respect to the local coordinates $(x, w, v^1, \dots, v^n)$ we get

$$\begin{aligned} \omega^1 &= \bar{f}\,dw, \\ \omega^2 &= \bar{A}\,dx + \bar{C}\,dw, \\ \tau^{\alpha+2} &= dv^\alpha + \bar{H}^\alpha\,dw, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \bar{A} &= \bar{c}, \\ \bar{f} &= \bar{M}/\bar{c}, \\ \bar{C} &= \sum_\alpha (\bar{b}_\alpha/\bar{c})v^\alpha + (\bar{b}/\bar{c}), \\ \bar{H}^\alpha &= \sum_\beta \bar{h}^\alpha_\beta v^\beta + \bar{h}^\alpha_0, \quad \bar{h}^\alpha_\beta + \bar{h}^\beta_\alpha = 0, \quad \text{and } \bar{G}_\alpha = 0. \end{aligned} \tag{4.16}$$

(Here, obviously, $\bar{A}$, $\bar{f}$, $\bar{c}$, $\bar{M}$ and $\bar{C}$ are the same as A, f, c, M and C, respectively.)

Next we will determine the functions $\bar{M}$, $\bar{c}$, $\bar{b}_\alpha$, $\bar{b}$, $\bar{h}^\alpha_\beta$ and $\bar{h}^\alpha_0$ such that the partial differential equations (A1)–(D3) are satisfied. Substituting in (A1)–(D3) from (4.6) and (4.16), we obtain

$$\begin{aligned} &(\bar{B})'_\alpha = 0, \quad \alpha = 1, \dots, n, \quad \text{where } \bar{B} \text{ is given by the analogue of (2.16)}, \\ &(\bar{R})'_\alpha = 0, \quad \alpha = 1, \dots, n, \quad \text{where } \bar{R} \text{ is given by the analogue of (2.20)}, \\ &(\bar{A}\bar{B})'_w + (\bar{R})'_x = -k\bar{A}\bar{f}, \\ &\bar{A}\bar{B}\bar{T}_\alpha = 0, \quad \alpha = 1, \dots, n, \\ &(\bar{T}_\alpha)'_x = 0, \quad \alpha = 1, \dots, n, \\ &(\bar{h}^\alpha_\beta)'_x = 0, \quad \alpha, \beta = 1, \dots, n. \end{aligned} \tag{4.17}$$

We get additional conditions if we express the equalities $\bar{B}_\alpha = 0$ (see (2.15)):

$$(\bar{h}_0^\alpha)'_x = -\bar{b}_\alpha, \quad \alpha = 1, \dots, n. \tag{4.18}$$

Now we shall prove

Proposition 4.1. *If $\bar{B} \neq 0$ at a regular point p, then the metric g is a product metric in a neighbourhood of p.*

Proof. From the fourth equation of (4.17) we see that $\bar{T}_\alpha = 0$ for all α, and hence,by (4.6), $\bar{b}_\alpha = 0$. From (4.16) we then derive that $\bar{C}$ depends only on x and w. The other conditions now mean

$$\begin{aligned} &(\bar{h}_\beta^\alpha)'_x = (\bar{h}_0^\alpha)'_x = 0, \\ &(\bar{A}\bar{B})'_w + (\bar{R})'_x = -k\bar{A}\bar{f}. \end{aligned} \tag{4.19}$$

The second equation ensures the curvature homogeneity. By changing the x-coordinate we can transform $\bar{A}$ to be 1. (In fact, we can take a new coordinate $\bar{x} = \bar{x}(x, w)$ as any function satisfying the partial differential equation $\frac{\partial \bar{x}}{\partial x} = \bar{A}$.) Then we get, in the new local coordinates,

$$\begin{aligned} &\bar{A} = 1, \\ &\bar{f} = \bar{f}(x, w), \\ &\bar{C} = \bar{C}(x, w), \\ &\bar{H}^\alpha = \sum_\beta \bar{h}_\beta^\alpha(w) v^\beta + \bar{h}_0^\alpha(w). \end{aligned} \tag{4.20}$$

Next we will annihilate the functions $\bar{H}^\alpha$ by an appropriate coordinate transformation. To do this, consider the differential equation for a $(n \times n)$-matrix function ψ

$$\frac{d\psi}{dw} = \psi \bar{H}, \tag{4.21}$$

where $\bar{H} = (\bar{h}_\beta^\alpha)$. Because $\bar{H}$ is skew-symmetric, we see easily that $\psi^t\psi$ is constant. By a proper choice of initial conditions we can assume that $\psi^t\psi = I$, i.e., ψ is an orthogonal matrix function depending only on w. We also put

$$\underline{\mu} = \int_{w_0}^{w} \psi \bar{H}_0 \, dw, \tag{4.22}$$

where $\bar{H}_0 = (\bar{h}_0^\alpha)$, a column matrix. $\underline{\mu}$ also depends only on w.

We now introduce new variables $(q^1, \dots, q^n)$ as follows:

$$\underline{q} = \psi \underline{v} + \underline{\mu}, \tag{4.23}$$

where $\underline{q}$ is the column vector (q^α). We have then

$$d\underline{q} = (d\psi)\underline{v} + \psi(d\underline{v}) + d\underline{\mu}, \tag{4.24}$$

where

$$d\psi = \psi \bar{H}\, dw, \quad d\underline{\mu} = \psi \bar{H}_0\, dw. \tag{4.25}$$

From (4.24) and (4.25) we get

$$d\underline{q} = \psi\big(d\underline{v} + (\bar{H}\,\underline{v} + \bar{H}_0)\, dw\big). \tag{4.26}$$

Comparing this last formula with (4.14), we get

$$d\underline{q} = \psi\underline{\tau}, \tag{4.27}$$

where $\underline{\tau} = (\tau^\alpha)$. Putting $\underline{\sigma} = \psi\underline{\tau}$ we obtain a new orthonormal coframe $(\omega^1, \omega^2, \sigma^{\alpha+2})$ such that in the coordinates (x, w, q^α) we have the following expressions:

$$\begin{aligned} \omega^1 &= \bar{f}(x,w)\, dw, \\ \omega^2 &= dx + \bar{C}(x,w)\, dw, \\ \sigma^{\alpha+2} &= dq^\alpha. \end{aligned} \tag{4.28}$$

From these formulas we see at once that we have a product metric in this case. This proves the proposition.

Consequently, *if* (M^{n+2}, g) *is locally irreducible, the function* $\bar{B}$ *must vanish at each* (*regular*) *point* $p \in M$. We assume $\bar{B} = 0$ in the sequel.

The conditions (4.17) then reduce to

$$\begin{aligned} (\bar{R})'_x &= -k\bar{A}\bar{f}, \\ (\bar{T}_\alpha)'_x &= 0, \\ (\bar{h}^\alpha_\beta)'_x &= 0, \\ (\bar{h}^\alpha_0)'_x &= -\bar{b}_\alpha. \end{aligned} \tag{4.29}$$

Moreover, we get additional conditions coming from $\bar{B} = 0$, namely (2.16) implies

$$(\bar{A})'_w - (\bar{C})'_x = 0$$

and hence,

$$(\bar{c})'_w - (\bar{b}/\bar{c})'_x = 0, \quad (\bar{b}_\alpha/\bar{c})'_x = 0. \tag{4.30}$$

By changing the x-coordinate we obtain, once again, $\bar{A} = \bar{c} = 1$. In the new local coordinates we get (changing a bit the notation of (4.16))

$$\begin{aligned} \bar{A} &= 1, \\ \bar{f} &= \bar{f}(x,w), \\ \bar{C} &= \sum_\alpha c_\alpha v^\alpha + c_0, \\ \bar{H}^\alpha &= \sum_\beta \bar{h}^\alpha_\beta v^\beta + \bar{h}^\alpha_0. \end{aligned} \tag{4.31}$$

Here the conditions (4.29)–(4.30) show that the functions c_α, c_0 and $\bar{h}^\alpha_\beta$ depend only on w, and that $\bar{h}^\alpha_0$ is of the form

$$\bar{h}^\alpha_0 = -c_\alpha(w)\,x + d^\alpha(w), \tag{4.32}$$

where d^α is an arbitrary function of one variable. Moreover, the conditions (2.14) and (2.20) show that $\bar{R} = \bar{f}'_x$.

The first condition of (4.29) can now be rewritten in the form

$$\bar{f}''_{xx} + k\bar{f} = 0. \tag{4.33}$$

(This condition ensures the curvature homogeneity.) So $\bar{f}$ is given by

$$\bar{f}(x,w) = a(w)\exp(\sqrt{-k}\,x) + b(w)\exp(-\sqrt{-k}\,x), \quad \text{if } k < 0 \tag{4.34}$$

or

$$\bar{f}(x,w) = a(w)\cos(\sqrt{k}\,x) + b(w)\sin(\sqrt{k}\,x), \quad \text{if } k > 0, \tag{4.35}$$

where $a(w)$ and $b(w)$ are differentiable functions such that $\bar{f}(x,w) \neq 0$.

We now introduce new variables $x^1, \ldots, x^{n+1}$ by

$$x^1 = x, \quad x^i = v^{i-1}, \quad i = 2, \ldots, n+1 \tag{4.36}$$

and we also put

$$\begin{aligned}
\bar{\omega}^0 &= \omega^1,\\
\bar{\omega}^1 &= \omega^2,\\
\bar{\omega}^i &= \omega^{i+1}, \quad i = 2, \ldots, n+1,
\end{aligned}$$

$$\begin{aligned}
l^1(w) &= c_0(w),\\
l^i(w) &= d^{i-1}(w), \quad i = 2, \ldots, n+1,
\end{aligned} \tag{4.37}$$

$$D = \begin{pmatrix} 0 & c_1 & c_2 & \ldots & c_n \\ -c_1 & 0 & \bar{h}^1_2 & \ldots & \bar{h}^1_n \\ -c_2 & \bar{h}^2_1 & 0 & \ldots & \bar{h}^2_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_n & \bar{h}^n_1 & \bar{h}^n_2 & \ldots & 0 \end{pmatrix}.$$

We see that D is a skew-symmetric matrix function of w only. Then we can write the following expressions for the coframe $(\bar{\omega}^0, \ldots, \bar{\omega}^{n+1})$:

$$\begin{aligned}
\bar{\omega}^0 &= \bar{f}(x^1, w)\,dw,\\
\bar{\omega}^i &= dx^i + \sum_{j=1}^{n+1} D^i_j x^j\,dw + l^i\,dw, \quad i = 1, \ldots, n+1.
\end{aligned} \tag{4.38}$$

We want to annihilate the functions $l^i(w)$ by a coordinate transformation of the form

$$\tilde{x}^i = x^i + k^i(w), \quad i = 1, \dots, n+1. \tag{4.39}$$

Taking the differential of (4.39) and substituting in (4.38), we see that the functions k^i must satisfy the system of ordinary differential equations (in matrix notation)

$$K' = -DK + L, \tag{4.40}$$

where $K = \left(k^i(w)\right)$, $K' = \left(k^{i\prime}(w)\right)$ and $L = (l^i)$. Any solution of (4.40) determines a new system of local coordinates (4.39) such that (4.38) is transformed to the form

$$\begin{aligned} \bar{\omega}^0 &= \bar{f}(w, \tilde{x}^1)\, dw, \\ \bar{\omega}^i &= d\tilde{x}^i + \sum_{j=1}^{n+1} D^i_j \tilde{x}^j\, dw, \end{aligned} \tag{4.41}$$

where $D = \left(D^i_j(w)\right)$ is a skew-symmetric matrix function of the variable w. Moreover, $\bar{f}$ satisfies (4.34) or (4.35).

In [6] the authors have proved that the metrics given by (4.41) and (4.34), or (4.35), respectively, are curvature homogeneous. Hence we have described all locally irreducible curvature homogeneous metrics around regular points with the curvature tensor of the type $H^2 \times \mathbb{R}^n$, or $S^2 \times \mathbb{R}^n$, respectively.

We can summarize (using also the structure theorem from the Introduction):

Theorem 4.2. *Let (M^{n+2}, g) be a locally non-homogeneous, locally irreducible, curvature homogeneous manifold with a symmetric model space. Then the model space is either $H^2(-\lambda^2) \times \mathbb{R}^n$ or $S^2(\lambda^2) \times \mathbb{R}^n$. Further, there exists a dense open subset U of M such that in a neighbourhood of every point $p \in U$ there exist local coordinates $(w, x^1, \dots, x^{n+1})$ around p and an orthonormal coframe of the form*

$$\begin{aligned} \omega^0 &= f(x^1, w)\, dw, \\ \omega^i &= dx^i + \sum_{j=1}^{n+1} D^i_j(w) x^j\, dw, \quad i = 1, \dots, n+1. \end{aligned}$$

Here $D = \left(D^i_j(w)\right)$ is a skew-symmetric matrix function of the variable w and the function $f \neq 0$ is given either by

$$f(x, w) = a(w)\exp(\lambda x) + b(w)\exp(-\lambda x),$$

or by

$$f(x, w) = a(w)\cos(\lambda x) + b(w)\sin(\lambda x)$$

respectively.

Remark 4.3. Conversely, any local metric g of the form given above is curvature homogeneous, with the symmetric model $H^2(-\lambda^2) \times \mathbb{R}^n$ or $S^2(\lambda^2) \times \mathbb{R}^n$. "Generically", these metrics are also non-homogeneous and locally irreducible (but not always, see [6] for more details).

Remark 4.4. The authors do not know any *geometrical* construction of the dense subset $U \subset M$ (of regular points). Anyway, such a dense subset can always be constructed, starting from a fixed coordinate atlas on M.

5. An extension: spaces with constant scalar curvature along the leaves and parabolically foliated spaces

Up to now, we have only considered curvature homogeneous spaces (M^{n+2}, g) with a symmetric model space (which are foliated by n-dimensional Euclidean leaves). For this case, the scalar curvature is constant. In this section, we will first extend the result obtained in Theorem 4.2 to the case where the scalar curvature is constant only along each fixed Euclidean leaf. Then we will show that this result leads to the explicit description of the semi-symmetric spaces which are parabolically foliated in the sense of Z.I. Szabó.

So, let (M^{n+2}, g) be a smooth $(n+2)$-dimensional foliated semi-symmetric space with constant scalar curvature along the Euclidean leaves. At every point the curvature tensor is then of the type $H^2 \times \mathbb{R}^n$ or $S^2 \times \mathbb{R}^n$. We know from Section 2 that we can find local coordinates $(w, x, y^1, \ldots, y^n)$ in a neighbourhood of each point p of M such that the metric is given in the standard form (2.1).

Let $2k(x, w)$ denote the scalar curvature. (It is independent of the variables y^α because we suppose it to be constant along the leaves.) Then we still have the formulas (2.3)–(2.6) for the connection forms, where now k is no longer a constant, but a function of w and x. (2.7) is then equivalent to

$$\frac{\partial}{\partial y^\alpha}(kAf) = 0. \tag{5.1}$$

As k depends only on x and w, we have that the same holds for the function Af. So, (2.10) continues to hold. We can now follow exactly the same procedure as in the curvature homogeneous case. The only difference is, that the condition (A3) (or, equivalently, the first condition of (4.29)), which in the curvature homogeneous case provided the differential equation (4.33) for f, is now only a formula for expressing the scalar curvature, namely

$$\mathrm{Sc}(g) = 2k = -2\bar{f}^{-1}\bar{f}''_{x^1x^1}.$$

We can therefore state the following theorem which modifies slightly Theorem 4.2:

Theorem 5.1. *Let (M^{n+2}, g) be a locally irreducible semi-symmetric space foliated by n-dimensional Euclidean leaves and such that its scalar curvature is constant along each leaf. Then there exists a dense open subset U of M such that in a neighbourhood of every point $p \in U$ there exist local coordinates $(w, x^1, \ldots, x^{n+1})$ and an orthonormal* *coframe of the form*

$$\omega^0 = f(w, x^1)\, dw,$$
$$\omega^i = dx^i + \sum_{j=1}^{n+1} D^i_j(w) x^j\, dw, \quad i = 1, \dots, n+1,$$

where $D^i_j(w) + D^j_i(w) = 0$. *The scalar curvature of this metric is given by*

$$\mathrm{Sc}(g) = -2f^{-1} f''_{x^1 x^1}.$$

It was proved in [6] that the local metrics above are semi-symmetric. Moreover, in [6] it is also shown that using an additional change of variables, one can write each of the metrics g from above in the form

$$g = \sum_{i=1}^{n+1} du^i \otimes du^i + f^2\big(w, \sum_{j=1}^{n+1} b_j(w)\, u^j\big)\, dw \otimes dw.$$

This shows that *the spaces considered in Theorem* 5.1 *are generalized warped products* (in the sense of K. Sekigawa [9]) *and are foliated by totally geodesic Euclidean leaves of codimension one.* (In the general situation of Proposition 2.1 the minimal codimension of the Euclidean leaves is two.)

Finally, we shall recall the notion of a parabolically foliated semi-symmetric space in the sense of Z.I. Szabó [13]. For this purpose, let (M^{n+2}, g) be an $(n+2)$-dimensional semi-symmetric space foliated by n-dimensional Euclidean leaves and with a metric of the form (2.1). Let $(E_1, \dots, E_{n+2})$ be the orthonormal frame which is dual to the coframe $(\omega^1, \dots, \omega^{n+2})$. Then $E_3, \dots, E_{n+2}$ span the tangent spaces to the Euclidean leaves and E_1, E_2 span their orthogonal complements. In [12, p. 548], Szabó defines the linear endomorphisms $\mathfrak{B}_\alpha$, $\alpha = 1, \dots, n$ of $\mathrm{span}\{E_1, E_2\}$ by

$$D_X E_{\alpha+2} = \mathfrak{B}_\alpha(X) + \text{ linear combination of } E_3, \dots, E_{n+2}.$$

In [13], he then defines a particular class of foliated semi-symmetric manifolds by requiring that $(\mathfrak{B}_\alpha)^2 = 0$ for all α and, at each point, at least one of the operators $\mathfrak{B}_\alpha$ is non-zero. He calls such spaces *parabolically foliated.*

Using the standard formulas $D_{E_j} E_i = \sum_k \omega_i^k(E_j)\, E_k$, we derive easily, using (2.13), that in our coordinate system the endomorphisms $\mathfrak{B}_\alpha$ have the following matrix form (with respect to E_1, E_2):

$$\begin{pmatrix} a^\alpha & b^\alpha \\ c^\alpha & e^\alpha \end{pmatrix}, \tag{5.2}$$

where

$$\begin{aligned} a^\alpha &= \psi A f'_\alpha, \quad b^\alpha = B_\alpha, \\ c^\alpha &= \psi(A C'_\alpha - C A'_\alpha) - B_\alpha, \quad e^\alpha = \psi f A'_\alpha. \end{aligned} \tag{5.3}$$

The assumption $(\mathfrak{B}_\alpha)^2 = 0$ is equivalent to the conditions

$$a^\alpha + e^\alpha = 0, \quad \text{and } (a^\alpha)^2 + b^\alpha c^\alpha = 0.$$

The first condition is equivalent to $(Af)'_\alpha = 0$. This implies, together with (5.1),

$$k'_\alpha = 0,$$

that is, the scalar curvature $\mathrm{Sc}(g) = 2k$, is independent of the variables y^α, and hence constant along each Euclidean leaf.

Conversely, suppose we have an irreducible foliated semi-symmetric space with constant scalar curvature along each Euclidean leaf. Then we have seen in the previous chapter that we can find local coordinates in a neighbourhood of each regular point such that the metric has the form (2.1), where the functions A, f, C and H^α are given by (4.31). According to (5.2) and using (5.3), we see that the operators $\mathfrak{B}_\alpha$ have, at any regular point, the matrix form

$$\begin{pmatrix} 0 & 0 \\ c_\alpha & 0 \end{pmatrix}.$$

Hence $(\mathfrak{B}_\alpha)^2 = 0$ for all α. Further, suppose that all $\mathfrak{B}_\alpha$ are zero on an open set. Then $c_\alpha = 0$ for all α and so C depends only on x and w on this set. Just as in Proposition 4.1, it is easy to see that the metric is a direct product on an open neighbourhood.

So we have proved the following

Theorem 5.2. *If a locally irreducible semi-symmetric manifold (M^{n+2}, g) which is foliated by n-dimensional Euclidean leaves, is parabolically foliated in the sense of Szabó, then it has constant scalar curvature along the Euclidean leaves. Conversely, if (M^{n+2}, g) has constant scalar curvature along the Euclidean leaves, it is parabolically foliated on a dense open subset.*

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